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A discrepancy principle for generalized local regularization of linear inverse problems

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Abstract. A modified version of the classical discrepancy principle is formulated for use with generalized local regularization operators of the form $\mathcal{R}_\alpha = (a_\alpha I + \mathcal{A}_\alpha)^{-1} T_\alpha$ for the approximate solution of linear inverse problems in Banach space with deterministically modeled noise. The choice of the local regularization parameter according to the *a posteriori* parameter selection strategy is shown to result in a class of convergent regularization methods and a general rate of convergence is provided. As an example, the theory is applied to establish convergence and convergence rates for approximations obtained using a zeroth-order local regularization scheme with the modified principle for solving Volterra convolution equations in $L^p(0,1),\ p \in (1,\infty)$. A numerical example is provided to illustrate the practical use and effectiveness of the method.

Keywords. Local regularization, discrepancy principle, ill-posed Volterra equations.

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1 Introduction and background

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $\mathcal{A} : X \mapsto Y$ a continuous linear operator with unbounded inverse (e.g., $\mathcal{N}(\mathcal{A}) = \{0\}$ and $R(\mathcal{A})$ not closed in Y). Consider solving

$$\mathcal{A}u = f \tag{1.1}$$

for $u \in X$, an ill-posed linear inverse problem for which a unique solution exists that fails to depend continuously on data $f \in R(\mathcal{A}) \subseteq Y$. In practice, one is provided only inexact measurement data or data corrupted by noise, thus solution of equation (1.1) requires a regularization method be implemented. We view the problem in the context of deterministically modeled data error and place focus on the application of continuous methods for which convergence can be verified.

A convergent regularization method is formulated in two parts. The first is a parameter dependent family $\{\mathcal{R}_\alpha\}_{\alpha>0}$ of continuous operators $\mathcal{R}_\alpha : Y \mapsto X$ that

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is used to approximate $\mathcal{A}^{-1}$ in the sense that for every $u \in X$,

$$\lim_{\alpha \to 0^+} \|\mathcal{R}_\alpha \mathcal{A} u - u\|_X = 0.$$

Any such $\mathcal{R}_\alpha$ is said to be a regularization operator for $\mathcal{A}^{-1}$.

The second part consists of a strategy for selecting the parameter α as a function of the noise level $\delta > 0$ and hence operators $\left\{\mathcal{R}_{\alpha(\delta)}\right\}_{\delta>0}$ so that for every $u \in X$,

$$\lim_{\delta \to 0} \alpha(\delta) = 0$$

and

$$\limsup_{\delta \to 0} \left\{ \left\|\mathcal{R}_{\alpha(\delta)} g - u\right\|_X \;\middle|\; g \in Y,\ \|\mathcal{A}u - g\|_Y \le \delta \right\} = 0.$$

Classical regularization methods for linear inverse problems are well-established when X and Y are Hilbert spaces. With such methods, regularization operators take the form

$$\mathcal{R}_\alpha = g_\alpha(\mathcal{A}^* \mathcal{A}) \mathcal{A}^*, \tag{1.2}$$

where $\mathcal{A}^* : Y \mapsto X$ denotes the Hilbert adjoint of $\mathcal{A}$, and for each $\alpha > 0$, the continuous function $g_\alpha : [0, \|A\|^2] \to \mathbb{R}$ satisfies certain properties [6,10]. For instance, the choice $g_\alpha(t) = (\alpha + t)^{-1}$ yields the Tikhonov regularization operator $\mathcal{R}_\alpha = (\alpha I + \mathcal{A}^*\mathcal{A})^{-1} \mathcal{A}^*$. [1]

In certain problems, computational efficiency can be improved if regularization methods are employed that do not require use of the adjoint operator $\mathcal{A}^*$ ([17]). One such example is Lavrent'ev or simplified regularization which involves operators of the form $\mathcal{R}_\alpha = (\alpha I + \mathcal{A})^{-1}$. This regularization is only known to be valid for narrow classes of operators $\mathcal{A}$ (e.g., $\mathcal{A}$ monotone).

The subject of this paper, local regularization, is another example. Local regularization operators in general take the form $\mathcal{R}_\alpha = (a_\alpha I + \mathcal{A}_\alpha)^{-1} T_\alpha$, and are valid for use with wider classes of operators $\mathcal{A}$ provided particular assumptions outlined in Section 2 are satisfied. It is the goal of this paper to provide a practical parameter selection strategy that produces a convergent local regularization method.

1.1 Parameter choice rules

We let $f \in R(\mathcal{A})$ represent the exact data and $\bar{u} \in X$ denote the corresponding solution to equation (1.1), i.e. $\mathcal{A}\bar{u} = f$. The noise level $\delta > 0$ in the given data is assumed known and the measured data f^δ belongs to $B_\delta(f)$, the closed ball of radius δ centered at f. For each $\alpha > 0$, u_α and u_α^δ are used to denote approximations

[1] Throughout, I denotes the identity operator on the space to be understood in the context.

constructed from the exact data and measured data respectively, i.e. $u_\alpha := \mathcal{R}_\alpha f$ and $u_\alpha^\delta := \mathcal{R}_\alpha f^\delta$, where $\{\mathcal{R}_\alpha\}_{\alpha>0}$ denotes a family of regularization operators for $\mathcal{A}^{-1}$. We denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators from X into Y with operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$ and write $\mathcal{L}(X)$ when $X = Y$.

If $\mathcal{R}_\alpha \in \mathcal{L}(Y,X)$, then

$$\left\| u_\alpha^\delta - \bar{u} \right\|_X \leq \|\mathcal{R}_\alpha\|_{\mathcal{L}(Y,X)} \delta + \|u_\alpha - \bar{u}\|_X \tag{1.3}$$

provides a simple albeit useful bound on the total error in approximating $\bar{u}$ by u_α^δ. As $\alpha \to 0$, the regularization error, $\|u_\alpha - \bar{u}\|_X$, tends to zero while unboundedness of $\mathcal{A}^{-1}$ leads to unboundedness of $\{\|\mathcal{R}_\alpha\|\}_{\alpha>0}$. Hence any choice of $\alpha = \alpha(\delta)$ made prior to the construction of an approximation and for which

$$\alpha(\delta) \to 0 \quad \text{and} \quad \delta \left\|\mathcal{R}_{\alpha(\delta)}\right\|_{\mathcal{L}(Y,X)} \to 0 \quad \text{as} \quad \delta \to 0$$

is an *a priori* strategy yielding a convergent method for solving (1.1). An optimal choice of $\alpha = \alpha(\delta)$ relies however on knowledge of a bound on $\|u_\alpha - \bar{u}\|_X$ which depends on smoothness properties of $\bar{u}$ that are typically unknown [24].

We focus instead on *a posteriori* rules, practical strategies for which selection of $\alpha = \alpha(\delta, f^\delta)$ is typically performed "online" i.e. u_α^δ is computed at decreasing values of α until some criteria are satisfied. The most well–known *a posteriori* rule is the classical discrepancy principle due to Morozov [16, 19, 21]. Based upon the heuristic that the method should not produce results more accurate than the error level in the given data, $\alpha = \alpha(\delta, f^\delta)$ is chosen to satisfy

$$\left\| \mathcal{A}u_\alpha^\delta - f^\delta \right\|_Y = \tau\delta, \tag{1.4}$$

for fixed $\tau > 1$.

Although popular, the discrepancy principle in (1.4) is not best suited for use with all regularization operators. For instance, when X and Y are Hilbert spaces, the rate of convergence obtained when paired with Tikhonov regularization under standard source conditions on $\bar{u}$ is not of optimal order [6]. Furthermore, convergence of Lavrent'ev regularization with the discrepancy principle (1.4) is not guaranteed as demonstrated in [11].

Modifications to (1.4) have been studied as viable alternatives that lead to convergent and order optimal methods. The modified discrepancy principle proposed in [25], known as Arcangeli's rule when $s = 1$ and $m = 1/2$, specifies that $\alpha = \alpha(\delta, f^\delta)$ be chosen to satisfy

$$\alpha^m \left\| \mathcal{A}u_\alpha^\delta - f^\delta \right\|_Y = \tau\delta^s \quad m, s > 0. \tag{1.5}$$

Note that when $s = 1$ and $\alpha \mapsto \left\| \mathcal{A} u_\alpha^\delta - f^\delta \right\|_Y$ is monotone, the small value of α selected with (1.5) always exceeds that given by (1.4).

The modified discrepancy principle in (1.5) and similar variants were originally studied to improve and optimize rates of convergence with Tikhonov regularization [4, 5, 9, 12, 25]. Convergence and optimal convergence rates were also established for Lavrent'ev (simplified) regularization paired with the modified principle in (1.5) [7, 8, 11, 20].

Numerous parameter selection strategies have been formulated, many for regularization operators of the form (1.2) under the requirement that X and Y are Hilbert spaces, and other heuristic or error-free strategies for which the parameter choice does not depend explicitly on the noise level in the data, see e.g. [13, 14]. *Local regularization* however is not based on spectral representations nor does it require underlying spaces to be Hilbert spaces, hence strategies reliant on these aspects are not considered nor are strategies for which convergence cannot be guaranteed [2]. The so-called Balancing Principle in [24] is a recently introduced adaptive selection strategy, and although quite general, relies upon monotonicity assumptions that need not hold for generalized local regularization operators. Adaptation of such a principle for use with local regularization is however the subject of on-going study. We refer the interested reader to [6] and the many references therein for more on parameter selection strategies.

1.2 Outline of the paper

In this paper we develop a theoretically-sound *a posteriori* parameter selection strategy based on (1.5) for the method of generalized local regularization. In Section 2, the generalized framework and main convergence results for local regularization operators defined in [2] are recalled for use in later sections. We define the modified discrepancy principle for selecting the generalized local regularization parameter, and establish sufficient conditions for convergence of the resulting method and for a general convergence rate. In Section 3, the theory is applied to establish convergence and specific convergence rates of a particular local regularization for solving Volterra convolution equations in $L^p(0,1)$, $p \in (1,\infty)$ in which the parameter is selected using the modified principle. In Section 4, a numerical example is included to illustrate the practical application of the method.

[2] Convergence of a method cannot be guaranteed if α does not depend explicitly on the deterministically modeled noise level δ [1].

2 A discrepancy principle for local regularization

A general definition of the local regularization operator is provided to fix notation and concepts used below (see [2] for greater detail). Henceforth take $X = Y$ and let $\|\cdot\| = \|\cdot\|_X$.

Definition 2.1. Let $\bar{\alpha} > 0$. For each $\alpha \in (0, \bar{\alpha}]$, let $(X_\alpha, \|\cdot\|_\alpha)$ be a Banach space and assume that the following hold:

A1. The "data sampling" operator $T_\alpha \in \mathcal{L}(X, X_\alpha)$ satisfies

$$\|T_\alpha g\|_\alpha \leq M_T \|g\|, \quad g \in X,$$

for $M_T > 0$ independent of $\alpha \in (0, \bar{\alpha}]$.

A2. The operator $T_\alpha \mathcal{A}$ may be decomposed as

$$T_\alpha \mathcal{A} = D_\alpha + \mathcal{A}_\alpha r_\alpha,$$

for $r_\alpha, D_\alpha \in \mathcal{L}(X, X_\alpha)$ and $\mathcal{A}_\alpha \in \mathcal{L}(X_\alpha)$, where for some $a_\alpha \neq 0$, the following hold.

(i) The operator $(a_\alpha I + \mathcal{A}_\alpha)$ has a bounded inverse on X_α.

(ii) The operator D_α is approximated by $a_\alpha r_\alpha$ in the sense that for every $u \in X$

$$\|(D_\alpha - a_\alpha r_\alpha)\, u\,\|_\alpha = o(c(\alpha)) \quad \text{as } \alpha \to 0^+, \tag{2.1}$$

where $c(\cdot) : (0, \infty) \mapsto \mathbb{R}_+$ satisfies

$$\left\|(a_\alpha I + \mathcal{A}_\alpha)^{-1}\right\|_{\mathcal{L}(X_\alpha)} \leq \frac{1}{c(\alpha)}, \tag{2.2}$$

and $c(\alpha) \to 0$ as $\alpha \to 0^+$.

Then $\mathcal{R}_\alpha \in \mathcal{L}(X, X_\alpha)$ defined by

$$\mathcal{R}_\alpha := (a_\alpha I + \mathcal{A}_\alpha)^{-1} T_\alpha, \tag{2.3}$$

is a (zeroth-order) **local regularization operator**.

Remark 2.2. In the case of X a Hilbert space, a natural question is whether the regularization operators associated with classical methods such Tikhonov regularization or Lavrent'ev regularization can be considered local regularization operators in the sense of this definition. The somewhat surprising answer is *no*.

For classical Tikhonov regularization, if we make the expected definitions

$$X_\alpha = X, \;\; r_\alpha = I, \;\; T_\alpha = \mathcal{A}^*, \;\; \mathcal{A}_\alpha = \mathcal{A}^*\mathcal{A}, \;\; a_\alpha = \alpha,$$

it follows that

$$c(\alpha) = \mathcal{O}(\alpha),$$

and

$$D_\alpha \equiv T_\alpha \mathcal{A} - \mathcal{A}_\alpha r_\alpha = 0,$$

the zero operator on X. Thus the only way that (2.1) can hold is if $u = 0$. The same conclusion follows for Lavrent'ev regularization (for suitable $\mathcal{A}$) with the choices

$$X_\alpha = X, \;\; r_\alpha = I, \;\; T_\alpha = I, \;\; \mathcal{A}_\alpha = \mathcal{A}, \;\; a_\alpha = \alpha. \tag{2.4}$$

Thus the construction of a class of local regularization operators takes some care, which we illustrate by example in Section 3 in the case of $\mathcal{A}$ a ν-smoothing Volterra operator (see also [2]).

Remark 2.3. In general r_α can be expected to be either a restriction or projection type of operator. The distinguishing characteristic of local regularization is not the presence of r_α but, as is discussed in Remark 2.2 above, the ability of the method to satisfy the special condition (2.1).

The main convergence results for the generalized version of local regularization outlined above (with *a priori* parameter selection) are summarized here.

Theorem 2.4. [2] *Let* $\{\mathcal{R}_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ *be a collection of local regularization operators*

(i) *Then* $\{\mathcal{R}_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ *is a family of regularization operators for* $\mathcal{A}^{-1}$ *in the sense that for every* $u \in X$*,*

$$\lim_{\alpha\to 0^+} \|\mathcal{R}_\alpha \mathcal{A} u - r_\alpha u\|_\alpha = 0,$$

with r_α *given in* **A2**. *Further, if for* $u \in X$ *and* $\delta > 0$, *any selection of* $\alpha \in (0, \bar{\alpha}]$, $\alpha = \alpha(\delta)$ *is made satisfying*

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta}{c(\alpha(\delta))} \to 0, \quad \text{as} \;\; \delta \to 0 \tag{2.5}$$

with $c(\cdot)$ *given in* **A2**, *it follows that*

$$\limsup_{\delta\to 0} \left\{ \left\|\mathcal{R}_{\alpha(\delta)} g - r_{\alpha(\delta)} u\right\|_{\alpha(\delta)} \;\middle|\; g \in B_\delta(\mathcal{A}u) \right\} = 0.$$

(ii) *If $u \in X$ satisfies conditions ensuring that*

$$\|(D_\alpha - a_\alpha r_\alpha)\, u\,\|_\alpha \le \omega(\alpha) c(\alpha) \tag{2.6}$$

for $\omega = \omega(\alpha, u) > 0$ defined for all α sufficiently small and $\omega(\alpha) \to 0$ as $\alpha \to 0^+$, then

$$\|\mathcal{R}_\alpha \mathcal{A} u - r_\alpha u\|_\alpha = \mathcal{O}(\omega(\alpha)) \quad \textit{as} \ \alpha \to 0^+.$$

Suppose the data $\{f^\delta\}_{\delta>0}$ are given where $f^\delta \in B_\delta(\mathcal{A}u)$. If $\alpha = \alpha(\delta)$ is selected so that (2.5) holds, it follows that

$$\left\|\mathcal{R}_{\alpha(\delta)} f^\delta - r_{\alpha(\delta)} u\right\|_{\alpha(\delta)} = \mathcal{O}\left(\frac{\delta}{c(\alpha(\delta))} + \omega(\alpha(\delta))\right) \to 0 \quad \textit{as} \ \delta \to 0. \tag{2.7}$$

Remark 2.5. If D is a subspace of X and the condition in **A2**(ii) holds only for $u \in D$ rather than for all $u \in X$, one still obtains the convergence results in Theorem 2.4 however only for $u \in D$. In this case, the collection $\{\mathcal{R}_\alpha\}_{\alpha \in (0,\bar{\alpha}]}$ would be a family of local regularization operators for $\mathcal{A}^{-1}$ relative to D, i.e. for every $u \in D$,

$$\lim_{\alpha \to 0^+} \|\mathcal{R}_\alpha \mathcal{A} u - r_\alpha u\|_\alpha = 0.$$

2.1 A modified discrepancy principle for local regularization

Throughout the remainder of Section 2, we assume $\bar{\alpha} > 0$ is fixed, and for each $\alpha \in (0, \bar{\alpha}]$, $\mathcal{R}_\alpha$ is a local regularization operator as in Definition 2.1. Once again, a non-zero $f \in R(\mathcal{A})$ represents exact data and $\bar{u} \in X$ denotes the corresponding solution to equation (1.1), i.e. $\mathcal{A}\bar{u} = f$.

Definition 2.5. Let $\tau > 1$ be fixed and let $b(\cdot) : [0, \bar{\alpha}] \mapsto \mathbb{R}_+$ denote a continuous, monotone increasing function that satisfies

$$b(0) = 0 \quad \text{and} \quad \lim_{\alpha \to 0^+} \frac{b(\alpha) a_\alpha}{c(\alpha)} = 0. \tag{2.8}$$

For each $\delta > 0$ and $f^\delta \in B_\delta(f)$, the **modified discrepancy principle for local regularization** is to choose the local regularization parameter α to be

$$\alpha_\star(\delta, f^\delta) := \inf\left\{\alpha \in (0, \bar{\alpha}] \;\middle|\; d(\alpha, f^\delta) = \tau\delta\right\}, \tag{2.9}$$

where the discrepancy functional $d : (0, \bar{\alpha}] \times X \mapsto \mathbb{R}_+$ is defined by

$$d(\alpha, g) := b(\alpha) \left\| \mathcal{A}_\alpha w_\alpha - T_\alpha g \right\|_\alpha , \tag{2.10}$$

for $\alpha \in (0, \bar{\alpha}]$, $g \in X$, and $w_\alpha = \mathcal{R}_\alpha g$.

In order that this principle be a well-defined *a posteriori* parameter strategy for the local regularization theory developed earlier, we require the following assumptions in addition to **A1–A2**.

D1. For each $g \in X$, the mapping $\alpha \mapsto \left\| \mathcal{A}_\alpha w_\alpha - T_\alpha g \right\|_\alpha = \left\| a_\alpha \mathcal{R}_\alpha g \right\|_\alpha$ depends continuously on $\alpha \in (0, \bar{\alpha}]$.

D2. There exist continuous, monotone increasing functions $a_i(\cdot), \lambda_i(\cdot) : [0, \bar{\alpha}] \mapsto \mathbb{R}_+$ such that $0 = a_i(0) = \lambda_i(0)$ for $i = 1, 2$, and $a_1(\alpha) \leq a_\alpha \leq a_2(\alpha)$ and $\lambda_1(\alpha) \leq c(\alpha) \leq \lambda_2(\alpha)$ for all $\alpha \in (0, \bar{\alpha}]$.

D3. There exist constants $K, L > 0$ such that $\|r_\alpha\|_{\mathcal{L}(X, X_\alpha)} \leq L$ and $\|\mathcal{A}_\alpha\|_{\mathcal{L}(X_\alpha)} \leq K$ for all $\alpha \in (0, \bar{\alpha}]$.

Remark 2.7. (i) A continuity assumption like **D1** is common in classical methods with *a posteriori* parameter selection. It is typically assumed that the map $\alpha \mapsto g_\alpha$ in (1.2) is continuous (c.f. pg 84 [6]).

(ii) The scalar functions a_α and $c(\alpha)$ need not be continuous nor monotone in contrast to their counterparts in Tikhonov and Lavrent'ev regularization in which the scalar term αI provides stability and $\|\mathcal{R}_\alpha\|_{\mathcal{L}(X)} \leq \frac{1}{\alpha}$. Condition **D2** serves as an analog of these properties.

(iii) From Remark 2.2 and the assignments in (2.4), the Lavrent'ev regularization operator $(\alpha I + \mathcal{A})^{-1}$ can be written in the form of (2.3) (although **A2**(ii) fails to hold). With $b(\alpha) = \alpha^m$, the principle (2.9) coincides with (1.5) when $s = 1$.

Continuity of the discrepancy in (2.10) as a function of α, attainability required in (2.9), and convergence of the method are direct consequences of the given assumptions, an appropriate (relative) scaling of the factors $b(\cdot)$ and τ appearing in the discrepancy principle, and an assumption that the method's sampled "signal" is greater than the level of noise (cf. (2.11) and (2.12), respectively).

Lemma 2.8. *Suppose that* **D1- D3** *hold and that the choice of* $b(\cdot)$ *in the discrepancy functional* (2.10) *is scaled with respect to* τ *so that*

$$\frac{b(\bar{\alpha})}{\tau} \geq \frac{a_2(\bar{\alpha}) + K}{a_1(\bar{\alpha})}. \tag{2.11}$$

Then for any $\delta > 0$ and for $f^\delta \in B_\delta(f)$ satisfying

$$\left\| T_{\bar\alpha} f^\delta \right\|_{\bar\alpha} > \delta, \tag{2.12}$$

there exists $\alpha_\star = \alpha_\star(\delta, f^\delta) > 0$ satisfying the discrepancy criterion (2.9).

Proof. Note first that for each f^δ, the mapping $\alpha \mapsto d(\cdot, f^\delta)$ is continuous on $(0, \bar\alpha]$. Indeed, given the form of $\mathcal{R}_\alpha$ in (2.3), we may express (2.10) as

$$d(\alpha, f^\delta) = b(\alpha) \left\| a_\alpha \mathcal{R}_\alpha f^\delta \right\|_\alpha = b(\alpha) a_\alpha \left\| u_\alpha^\delta \right\|_\alpha, \tag{2.13}$$

thus continuity is an immediate consequence of **D1**.

From (2.2), (2.3), **A1**, **D2** and **D3**, we have

$$\left\| T_\alpha f^\delta \right\|_\alpha \leq (a_2(\bar\alpha) + K) \left\| u_\alpha^\delta \right\|_\alpha \leq (a_2(\bar\alpha) + K) \frac{M_T \left\| f^\delta \right\|}{c(\alpha)}$$

and therefore

$$B(\alpha) \left\| T_\alpha f^\delta \right\|_\alpha \leq d(\alpha, f^\delta) \leq \frac{M_T b(\alpha) a_\alpha}{c(\alpha)} \left\| f^\delta \right\|, \tag{2.14}$$

where we define $B(\cdot) : [0, \bar\alpha] \mapsto \mathbb{R}_+$ by

$$B(\alpha) := \frac{b(\alpha) a_1(\alpha)}{a_2(\bar\alpha) + K}. \tag{2.15}$$

Note that B is monotonically increasing and continuous with $B(0) = 0$.

It follows from (2.11), (2.12) and (2.14) that $d(\bar\alpha, f^\delta) \geq \tau\delta$, and from (2.8) and (2.14) that $\lim_{\alpha \to 0^+} d(\alpha, f^\delta) = 0$. With the continuity of d, we conclude that there exists an $\alpha_\star \in (0, \bar\alpha]$ for which $d(\alpha_\star, f^\delta) = \tau\delta$, i.e. the infimum in (2.9) is attained. □

To obtain existence of $\alpha_\star(\delta, f^\delta)$ for all $\delta > 0$ sufficiently small and all $f^\delta \in B_\delta(f)$, $f \neq 0$, an additional assumption (2.16) on $T_\alpha f$ is required, one which is satisfied naturally in the case that $\|T_\alpha f\|_\alpha \to c\|f\|$ as $\alpha \to 0$ for some $c = c(f) > 0$ and for $\bar\alpha = \bar\alpha(f) > 0$ sufficiently small.

Theorem 2.9. *Suppose that* **D1- D3** *hold and $b(\cdot)$ is scaled with respect to τ so that (2.11) holds. Assume further that there exists an $\epsilon = \epsilon(f) > 0$ for which*

$$\|T_\alpha f\|_\alpha \geq \epsilon \quad \textit{for all} \quad \alpha \in (0, \bar\alpha]. \tag{2.16}$$

Then the following conclusions hold.

(i) *For all $\delta > 0$ sufficiently small and for any $f^\delta \in B_\delta(f)$ there exists $\alpha_\star = \alpha_\star(\delta, f^\delta) > 0$ satisfying the discrepancy principle* (2.9)*. Further,*

$$\alpha_\star(\delta, f^\delta) \to 0 \quad \text{as} \quad \delta \to 0, \tag{2.17}$$

and

$$\left\| u^\delta_{\alpha_\star} - r_{\alpha_\star}\bar{u} \right\|_{\alpha_\star} \to 0 \quad \text{as} \quad \delta \to 0. \tag{2.18}$$

(ii) *Suppose $u = \bar{u}$ satisfies conditions ensuring that* (2.6) *holds for some monotone increasing function $\omega(\cdot) : [0, \infty) \mapsto \mathbb{R}_+$ where $\omega(\alpha) \to 0$ as $\alpha \to 0^+$. Then $\alpha_\star = \alpha_\star(\delta, f^\delta) \in (0, \bar{\alpha}]$ selected according to* (2.9) *yields a rate of convergence*

$$\left\| u^\delta_{\alpha_\star} - r_{\alpha_\star}\bar{u} \right\|_{\alpha_\star} = \mathcal{O}\left(\frac{M_T \delta}{\lambda_1((b \cdot a_2)^{-1}(\delta))} + \omega((b \cdot a_1)^{-1}(\delta)) \right) \tag{2.19}$$

as $\delta \to 0$.

Proof. (i) Let $\delta \in (0, \epsilon/(M_T + 1))$ and pick arbitrary $f^\delta \in B_\delta(f)$. Then

$$\left\| T_{\bar{\alpha}} f^\delta \right\|_{\bar{\alpha}} \geq \| T_{\bar{\alpha}} f \|_{\bar{\alpha}} - \left\| T_{\bar{\alpha}} \left(f^\delta - f \right) \right\|_{\bar{\alpha}} \geq \epsilon - M_T \delta > \delta,$$

so that f^δ satisfies the hypothesis (2.12) of Lemma 2.8. It then follows that for all $\delta > 0$ sufficiently small and for any $f^\delta \in B_\delta(f)$ there exists $\alpha_\star(\delta, f^\delta) > 0$ satisfying (2.9).

We next prove $\lim\limits_{\delta \to 0} \alpha_\star(\delta, f^\delta) = 0$. To simplify notation, let $\alpha = \alpha_\star(\delta, f^\delta)$. Then for $B(\cdot)$ defined in (2.15),

$$\begin{aligned} \tau\delta &= d(\alpha, f^\delta) \\ &\geq B(\alpha) \left(\| T_\alpha f \|_\alpha - \left\| T_\alpha (f^\delta - f) \right\|_\alpha \right) \\ &\geq B(\alpha) \left(\epsilon - M_T \delta \right) \end{aligned} \tag{2.20}$$

so that

$$\left(\tau + B(\bar{\alpha}) M_T \right) \delta \geq \epsilon B(\alpha).$$

That is,

$$G_1 \delta \geq (b \cdot a_1)(\alpha), \tag{2.21}$$

where $G_1 = \left(\tau + B(\bar{\alpha}) M_T \right) \left(a_2(\bar{\alpha}) + K \right) / \epsilon > 0$, with which we may conclude from the monotonicity of b, a_1 that $\alpha \leq (b \cdot a_1)^{-1} (G_1 \delta)$, establishing the convergence in (2.17).

To obtain the convergence in (2.18), use that

$$\left\|u_\alpha^\delta - u_\alpha\right\|_\alpha \leq \frac{1}{c(\alpha)} \left\|T_\alpha f^\delta - T_\alpha f\right\|_\alpha \leq \frac{M_T \delta}{c(\alpha)}, \tag{2.22}$$

$$\begin{aligned} \left\|u_\alpha^\delta\right\|_\alpha &\leq \left\|u_\alpha^\delta - u_\alpha\right\|_\alpha + \|u_\alpha - r_\alpha \bar{u}\|_\alpha + \|r_\alpha \bar{u}\|_\alpha \\ &\leq \frac{M_T}{\tau} \frac{b(\alpha) a_\alpha}{c(\alpha)} \left\|u_\alpha^\delta\right\|_\alpha + \|u_\alpha - r_\alpha \bar{u}\|_\alpha + L \|\bar{u}\|, \end{aligned} \tag{2.23}$$

using (2.13), and hence

$$\left[1 - \frac{M_T}{\tau} \frac{b(\alpha) a_\alpha}{c(\alpha)}\right] \left\|u_\alpha^\delta\right\|_\alpha \leq \|u_\alpha - r_\alpha \bar{u}\|_\alpha + L \|\bar{u}\|. \tag{2.24}$$

With (2.17), it follows from (2.8) that

$$\lim_{\delta \to 0} \frac{M_T}{\tau} \frac{b(\alpha) a_\alpha}{c(\alpha)} = 0, \tag{2.25}$$

thus returning to (2.24),

$$\limsup_{\delta \to 0} \left\|u_\alpha^\delta\right\|_\alpha \leq L \|\bar{u}\|, \tag{2.26}$$

by Theorem 2.4. Therefore

$$\begin{aligned} \lim_{\delta \to 0} \left\|u_\alpha^\delta - r_\alpha \bar{u}\right\|_\alpha &\leq \limsup_{\delta \to 0} \left(\frac{M_T}{\tau} \frac{b(\alpha) a_\alpha}{c(\alpha)} \left\|u_\alpha^\delta\right\|_\alpha + \|u_\alpha - r_\alpha \bar{u}\|_\alpha \right) \\ &= 0. \end{aligned}$$

(ii) It was shown in Theorem 2.1 of [2] that

$$\|u_\alpha - r_\alpha \bar{u}\|_\alpha \leq \omega(\alpha) \tag{2.27}$$

follows from (2.6), so using the monotonicity of ω with (2.21),

$$\|u_\alpha - r_\alpha \bar{u}\|_\alpha = \mathcal{O}\left(\omega\left((b \cdot a_1)^{-1}(G_1 \delta)\right)\right). \tag{2.28}$$

On the other hand with (2.13), (2.20), and (2.24)–(2.27), we have

$$\begin{aligned} \delta &= \frac{b(\alpha) a_\alpha}{\tau} \left\|u_\alpha^\delta\right\|_\alpha \\ &\leq \frac{b(\alpha) a_2(\alpha)}{\tau \left(1 - \frac{M_T b(\alpha) a_\alpha}{\tau c(\alpha)}\right)} \left(\omega(\bar{\alpha}) + L \|\bar{u}\|\right) \\ &\leq \frac{(b \cdot a_2)(\alpha)}{G_2}, \end{aligned} \tag{2.29}$$

for some $G_2 > 0$ and all α sufficiently small.

Return to (2.22) with **D2** and the monotonicity of λ_1, a_2, and b to obtain

$$\left\| u_\alpha^\delta - u_\alpha \right\|_\alpha = \mathcal{O}\left(\frac{M_T \delta}{\lambda_1((b \cdot a_2)^{-1}(G_2\delta))} \right),$$

as $\delta \to 0$, which together with (2.28) yields the rate stated in (2.19). □

It follows from Theorem 2.9 that the modified discrepancy principle for local regularization behaves like the *a priori* parameter choice rule in Theorem 2.4.

Corollary 2.10. *Under the assumptions of Theorem* 2.9*(i), there exists* $G_1 > 0$ *so that*

$$\alpha_\star(\delta, f^\delta) \le (b \cdot a_1)^{-1} (G_1\delta) \tag{2.30}$$

for all $\delta > 0$ *sufficiently small and any* $f^\delta \in B_\delta(f)$*. Further,*

$$\frac{\delta}{c\,(\alpha_\star(\delta, f^\delta))} \to 0 \quad as \quad \delta \to 0. \tag{2.31}$$

If in addition $\bar{u}$ *satisfies the conditions in Theorem* 2.9*(ii), then there exists* $G_2 > 0$ *so that*

$$\alpha_\star(\delta, f^\delta) \ge (b \cdot a_2)^{-1}(G_2\delta) \tag{2.32}$$

for all $\delta > 0$ *sufficiently small and any* $f^\delta \in B_\delta(f)$*.*

Proof. The upper and lower bounds on $\alpha_\star$ in (2.30) and (2.32) are obtained from (2.21) and (2.29), respectively, from the proof of Theorem 2.9 along with the monotonicity properties of b, a_1, and a_2. In addition, note that for $\alpha = \alpha_\star(\delta, f^\delta)$,

$$\frac{\delta}{c(\alpha)} = \frac{1}{\tau} d(\alpha, f^\delta) = \frac{1}{\tau} \frac{b(\alpha)}{c(\alpha)} a_\alpha \left\| u_\alpha^\delta \right\|_\alpha$$

so that the desired convergence in (2.31) follows from (2.8), (2.17), and (2.26). □

3 Application to the ν–smoothing Volterra problem in $L^p(0, 1)$

We now apply the generalized theory to establish convergence of a local regularization method involving the modified discrepancy principle defined in Section 2.1.

In this section, X denotes the Lebesgue space $L^p(0, 1)$ for fixed $p \in (1, \infty)$ with usual norm $\|\cdot\|$. We define $\mathcal{A} \in \mathcal{L}(X)$ by

$$\mathcal{A}u(t) := \int_0^t k(t - s)u(s)\, ds, \qquad a.e. \;\; t \in (0, 1) \tag{3.1}$$

with kernel $k \in C^{\nu}[0,1]$ for fixed $\nu \in \mathbb{N}$, and

$$k^{(\ell)}(0) = 0, \quad \ell = 0, 1, ..., \nu - 2, \qquad \text{and} \qquad k^{(\nu-1)}(0) \neq 0,$$

in the case of $\nu \geq 2$, while $k(0) \neq 0$ in the case of $\nu = 1$. The ν–smoothing Volterra problem, solving (1.1) with $\mathcal{A}$ defined in (3.1), is a generalization of obtaining the νth derivative of a given function f and arises in applications such as population dynamics and mechanics [3, 26]. Note that the operator $\mathcal{A}$ is compact and injective with non-closed range, thus solving the ν–smoothing Volterra problem with inexact data warrants the use of a regularization method.

Local regularization of the ν–smoothing problem with *a priori* parameter choice was treated in [15, 16, 22] with $X = C[0,1]$ and in [2] with $X = L^p(0,1), 1 \leq p \leq \infty$. It is well-established that local regularization methods applied to Volterra problems lead to regularized equations that are still Volterra and hence the causal nature of the original problem remains intact (see e.g. [17]). Discretizations of the local regularized equations lead to lower triangular linear systems that can be solved sequentially thus numerical solution is faster and more efficient. This is in opposition to classical methods, such as Tikhonov regularization, which lead to the costly solution of full linear systems.

3.1 A local regularization scheme

We first introduce a particular family of local regularization operators for $\mathcal{A}^{-1}$ from [2] according to Definition 2.1 and fix these definitions for the remainder of the paper.

Fix $\bar{\alpha} > 0$. For each $\alpha \in (0, \bar{\alpha}]$, define

$$X_\alpha := L^p(0, 1 - \alpha), \tag{3.2}$$

the Lebesgue space with the usual norm $\|\cdot\|_{L^p(0,1-\alpha)}$. Define $r_\alpha : X \mapsto X_\alpha$ to be the restriction operator, i.e. for every $g \in X$,

$$r_\alpha g(t) := g(t), \qquad \text{a.e. } t \in (0, 1 - \alpha). \tag{3.3}$$

Let $\mathcal{B}_{\bar{\alpha}}$ denote the σ-algebra of Borel subsets of $[0, \bar{\alpha}]$. The set of all finite signed measures on $\mathcal{B}_{\bar{\alpha}}$ is denoted by $\mathcal{M}$ and may be equipped with the *variation norm*, $|||\cdot|||$, i.e.

$$|||\mu||| = |\mu|([0, \bar{\alpha}]) = \mu^+([0, \bar{\alpha}]) + \mu^-([0, \bar{\alpha}]), \quad \mu \in \mathcal{M},$$

where μ^+ and μ^- denote the positive and negative variations of μ respectively. Note that $(\mathcal{M}, |||\cdot|||)$ forms a Banach space [23].

Definition 3.1. [2] A collection of measures $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]} \subseteq \mathcal{M}$ is said to be a **local-regularizing family of measures** if it satisfies the following properties:

(i) There exists a $\sigma \in \mathbb{R}$ such that for each $j = 0, 1, ..., \nu$,

$$\int_{[0,\alpha]} \rho^j d\eta_\alpha(\rho) = \alpha^{j+\sigma} c_j \left(1 + C_j(\alpha)\right) \text{ for all } \alpha \in (0, \bar{\alpha}],$$

where

a. $C_j(\alpha)$ is a function for which there is a constant $\bar{C}_j \geq 0$

$$|C_j(\alpha)| \leq \bar{C}_j \alpha < 1 \text{ for all } \alpha \in (0, \bar{\alpha}];$$

b. the constants $c_0, c_1, \ldots, c_\nu \in \mathbb{R}$ and $c_\nu \neq 0$ are such that the roots of the polynomial $p_\nu(\lambda)$, defined by

$$p_\nu(\lambda) = \frac{c_\nu}{\nu!}\lambda^\nu + \frac{c_{\nu-1}}{(\nu-1)!}\lambda^{\nu-1} + ... + \frac{c_1}{1!}\lambda + \frac{c_0}{0!}, \tag{3.4}$$

have negative real part.

(ii) There exists a constant $\tilde{C} > 0$ such that for every $\alpha \in (0, \bar{\alpha}]$,

$$|\eta_\alpha|([0,\alpha]) \leq \tilde{C}\alpha^\sigma.$$

Remark 3.2. A large class of measures can be constructed satisfying conditions (i) and (ii); see, for example, the measures defined in Proposition 3.5 to follow and [15]. As is illustrated in [22], assumption (i)b is a stability condition needed to establish that the sampling operator T_α, defined in (3.5) below as an η_α-weighted averaging operator, leads to a well-posed construction of $\mathcal{R}_\alpha$.

Let $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]} \subseteq \mathcal{M}$ be a local-regularizing family of measures. For each $\alpha \in (0, \bar{\alpha}]$ and every $g \in X$, define[3]

$$T_\alpha g(t) := \frac{1}{\eta_\alpha([0,\alpha])} \int_{[0,\alpha]} g(t+\rho) d\eta_\alpha(\rho), \qquad \text{a.e. } t \in (0, 1-\alpha), \tag{3.5}$$

$$\mathcal{A}_\alpha r_\alpha g(t) := \frac{1}{\eta_\alpha([0,\alpha])} \int_0^t \int_{[0,\alpha]} k(t+\rho-s) d\eta_\alpha(\rho)\, g(s)\, ds \tag{3.6}$$

$$\mathcal{D}_\alpha g(t) := \frac{1}{\eta_\alpha([0,\alpha])} \int_{[0,\alpha]} \int_0^\rho k(\rho-s) g(t+s) ds\, d\eta_\alpha(\rho)$$

[3] Throughout, the Lebesgue measure $dm(s)$ is denoted by ds.

for a.e. $t \in (0, 1-\alpha)$, and

$$a_\alpha := \frac{1}{\eta_\alpha([0,\alpha])} \int_{[0,\alpha]} \int_0^\rho k(\rho - s) ds \, d\eta_\alpha(\rho). \tag{3.7}$$

From Definition 3.1, it follows that $c_0 > 0$ [2] and

$$\eta_\alpha([0,\alpha]) \geq \alpha^\sigma c_0 \left(1 - \bar{C}_0 \bar{\alpha}\right) > 0 \tag{3.8}$$

for all $\alpha \in (0, \bar{\alpha}]$, and that **A1** in Definition 2.1 is satisfied with

$$M_T = \frac{\tilde{C}}{c_0(1 - \bar{C}_0\bar{\alpha})}. \tag{3.9}$$

It also follows that **A2** in Definition 2.1 is satisfied with

$$c(\alpha) = C a_\alpha \tag{3.10}$$

for $C > 0$ (independent of α) and $\bar{\alpha}$ sufficiently small. Convergence of the method with *a priori* parameter selection follows from Theorem 2.4(i).

Under additional source conditions, such as $\bar{u} \in D(\mu)$, where for $\mu \in (0, \nu+1]$,

$$D(\mu) = \left\{ u = \frac{1}{\Gamma(\mu)} \int_0^{\cdot} (\cdot - s)^{\mu-1} w(s) \, ds, \quad w \in C[0,1] \right\}, \tag{3.11}$$

with Γ the usual Gamma function, a convergence rate is obtained as in Theorem 2.4(ii) with rate function given by

$$\omega(\alpha) = \mathcal{O}\left(\alpha^\mu\right). \tag{3.12}$$

Proofs of the above results are found in [2].

3.2 Convergence with the modified discrepancy principle

Henceforth let $\{\eta_\alpha\}_{\alpha \in (0,\bar{\alpha}]} \subseteq \mathcal{M}$ be a local-regularizing family of measures and $\{\mathcal{R}_\alpha\}_{\alpha \in (0,\bar{\alpha}]}$ the corresponding local regularization operators in (2.3) with the definitions given in Section 3.1. [4]

We now show, as an immediate consequence of Theorem 2.9, that the above local regularization scheme paired with the modified discrepancy principle in (2.9) results in a convergent local regularization method for the ν–smoothing problem. Under additional conditions on $\bar{u}$, we also obtain an *a posteriori* rate of convergence.

[4] We drop the use of the restriction operator r_α where no confusion exists.

Lemma 3.3. (i) *For $\bar{\alpha}$ sufficiently small, there exist constants* $0 < \kappa_1 < \kappa_2$ *for which* **D2** *is satisfied with*

$$a_i(\alpha) = \lambda_i(\alpha) = \kappa_i \alpha^\nu, \; i = 1, 2,$$

for all $\alpha \in (0, \bar{\alpha}]$.

(ii) *Condition* **D3** *is satisfied.*

Proof. We shall only prove part 2. The result in part 1 is well-known, a proof of which can be found in e.g. Lemma 3.2 of [2].

We claim that **D3** holds with $K = M_T \|k\|_{C[0,1]}$ and $L = 1$. Fix $\alpha \in (0, \bar{\alpha}]$. Then

$$\begin{aligned} \|\mathcal{A}_\alpha\|_{\mathcal{L}(L^p(0,1-\alpha))} &= \sup_{\|h\|_{L^p(0,1-\alpha)}=1} \|T_\alpha k * h\|_{L^p(0,1-\alpha)} \\ &\le \sup_{\|h\|_{L^p(0,1-\alpha)}=1} \|T_\alpha k\|_{L^\infty(0,1-\alpha)} \|h\|_{L^p(0,1-\alpha)} \\ &\le M_T \|k\|_{C[0,1]} \end{aligned}$$

using Young's theorem for convolutions. It is evident that **D3** holds with $L = 1$ from the observation $\|r_\alpha g\|_{L^p(0,1-\alpha)} \le \|g\|$ for all $g \in X$. □

In order to establish continuity of d in (2.10) and verify condition **D1**, we refine the choice of local-regularizing measures to those that are continuous in $\mathcal{M}$ with respect to the variation norm.

Definition 3.4. A collection of local-regularizing measures $\{\eta_\alpha\}_{\alpha \in (0,\bar{\alpha}]} \subseteq \mathcal{M}$ is said to be **continuous** if the map $\alpha \mapsto \eta_\alpha$ is continuous.

An example is provided below of one of the many families of continuous local-regularizing measures (c.f. Lemma 2.2 of [15], Proposition 3.2 of [2].)

Proposition 3.5. *There exists* $\psi \in L^q(0,1)$, *where* $1/p + 1/q = 1$, *such that the local-regularizing collection of measures* $\{\eta_\alpha\}_{\alpha \in (0,\bar{\alpha}]}$ *defined for each* $\alpha \in (0, \bar{\alpha}]$ *by*

$$d\eta_\alpha = \psi_\alpha d\rho, \tag{3.13}$$

is continuous, where

$$\psi_\alpha(\rho) = \begin{cases} \psi\left(\frac{\rho}{\alpha}\right), & \text{a.e. } \rho \in [0, \alpha], \\ \\ 0 & \text{a.e. } \rho \in (\alpha, \bar{\alpha}] \end{cases}. \tag{3.14}$$

Proof. Existence of a polynomial function ψ in (3.14) leading to a local-regularizing family of measures on $[0,\bar{\alpha}]$ defined according to (3.13) follows directly from the arguments in [2, 15]. Let $\psi \in L^q(0,1)$ denote any such function.

Fix $\alpha \in (0,\bar{\alpha})$ and let $h>0$ be such that $(\alpha+h) \in (0,\bar{\alpha}]$. To prove continuity from the right, use a change of variables and Hölder's inequality to obtain

$$
\begin{aligned}
&|||\eta_{\alpha+h}-\eta_\alpha|||\\
&= |\eta_{\alpha+h}-\eta_\alpha|\,([0,\alpha]) + |\eta_{\alpha+h}-\eta_\alpha|\,((\alpha,\alpha+h)) + |\eta_{\alpha+h}-\eta_\alpha|([\alpha+h,\bar{\alpha}])\\
&\le \int_{[0,\alpha]} \left|\psi\left(\frac{\rho}{\alpha+h}\right) - \psi\left(\frac{\rho}{\alpha}\right)\right| d\rho + \int_{(\alpha,\alpha+h)} \left|\psi\left(\frac{\rho}{\alpha+h}\right)\right| d\rho\\
&\le \alpha \left\|\psi\left(\left(1-\frac{|h|}{\alpha+h}\right)\cdot\right) - \psi(\cdot)\right\|_{L^q(0,1)} + \|\psi\|_{L^q(0,1)}\,|h|^{1/p}\,(\alpha+h)^{1/q},
\end{aligned}
$$

which goes to zero as $h \to 0$ by continuity of translations in $L^q(0,1)$, $q \in [1,\infty)$. Continuity from the left at each $\alpha \in (0,\bar{\alpha}]$ is established by reversing the roles of α and $\alpha+h$ in the above arguments with $h<0$ such that $(\alpha+h) \in (0,\bar{\alpha})$. □

Lemma 3.6. *Let $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]} \subseteq \mathcal{M}$ be a continuous local-regularizing family of measures. If for every $\alpha \in (0,\bar{\alpha}]$, η_α is concentrated on $[0,\alpha)$ or $\eta_\alpha(\{\alpha\})=0$, then for $\bar{\alpha}$ sufficiently small,* **D1** *is satisfied.* [5]

Proof. Fix $\alpha \in (0,\bar{\alpha})$ and let $h>0$ be such that $(\alpha+h) \in (0,\bar{\alpha}]$. Let $g \in X$. To prove continuity in **D1** from the right, first use a variation of constants formula to express

$$\mathcal{R}_\alpha g = \frac{T_\alpha g}{a_\alpha} - \mathcal{X}_\alpha * \frac{T_\alpha g}{a_\alpha},$$

in terms of the resolvent, $\mathcal{X}_\alpha \in L^1(0,1-\alpha)$, the unique function satisfying

$$\mathcal{X}_\alpha(t) + \int_0^t \frac{T_\alpha k(t-s)}{a_\alpha}\mathcal{X}_\alpha(s)ds = \frac{T_\alpha k(t)}{a_\alpha}, \quad \text{a.e. } t \in (0,1-\alpha), \tag{3.15}$$

for $\bar{\alpha}>0$ sufficiently small [3]. Note that $\mathcal{X}_\alpha \in C[0,1-\alpha]$ as $T_\alpha k \in C^\nu[0,1-\alpha]$. Then for all $t \in (0,1-(\alpha+h))$, we may define

$$\bar{\mathcal{X}}_h(t) := \mathcal{X}_{\alpha+h}(t) - \mathcal{X}_\alpha(t)$$

and

$$k_h(t) := \frac{T_{\alpha+h}k(t)}{a_{\alpha+h}} - \frac{T_\alpha k(t)}{a_\alpha}.$$

[5] As usual, a measure $\mu \in \mathcal{M}$ is said to be *concentrated* on a set E if $\mu(F)=0$ whenever $E \cap F = \emptyset$ for any set $F \subseteq [0,\bar{\alpha}]$.

Therefore

$$\|a_{\alpha+h}\ \mathcal{R}_{\alpha+h}g\|_{L^p(0,1-(\alpha+h))} - \|a_\alpha \mathcal{R}_\alpha g\|_{L^p(0,1-\alpha)} \tag{3.16}$$

$$\begin{aligned}
&\leq \|T_{\alpha+h}g - \mathcal{X}_{\alpha+h} * T_{\alpha+h}g - T_\alpha g + \mathcal{X}_\alpha * T_\alpha g\|_{L^p(0,1-(\alpha+h))}\\
&\leq \|T_{\alpha+h}g - T_\alpha g\|_{L^p(0,1-(\alpha+h))} + \|(\mathcal{X}_{\alpha+h} - \mathcal{X}_\alpha) * T_{\alpha+h}g\|_{L^p(0,1-(\alpha+h))}\\
&\qquad + \|\mathcal{X}_\alpha * (T_{\alpha+h}g - T_\alpha g)\|_{L^p(0,1-(\alpha+h))}\\
&\leq \frac{1}{C}\|T_{\alpha+h}g - T_\alpha g\|_{L^p(0,1-(\alpha+h))} + \left\|\bar{\mathcal{X}}_h\right\|_{L^1(0,1-(\alpha+h))} M_T \|g\|,
\end{aligned} \tag{3.17}$$

where in [2] it was established that for $\bar{\alpha}$ sufficiently small and all $\alpha \in (0,\bar{\alpha}]$,

$$1 + \|\mathcal{X}_\alpha\|_{L^1(0,1-\alpha)} \leq \frac{1}{C}, \tag{3.18}$$

with $C > 0$ the constant appearing in (3.10). Then by (3.15),

$$\bar{\mathcal{X}}_h(t) + \frac{T_{\alpha+h}k}{a_{\alpha+h}} * \bar{\mathcal{X}}_h(t) + k_h(t) * \mathcal{X}_\alpha(t) = k_h(t),$$

for $t \in (0, 1-(\alpha+h))$ and

$$\begin{aligned}
\left|\bar{\mathcal{X}}_h(t)\right| &\leq \int_0^t \left|\frac{T_{\alpha+h}k(t-s)}{a_{\alpha+h}}\right| \left|\bar{\mathcal{X}}_h(s)\right| ds + \left|\int_0^t k_h(t-s)\mathcal{X}_\alpha(s)ds\right| + |k_h(t)|\\
&\leq \left\|\frac{T_{\alpha+h}k}{a_{\alpha+h}}\right\|_{L^\infty(0,1-(\alpha+h))} \int_0^t \left|\bar{\mathcal{X}}_h(s)\right| ds\\
&\qquad + \|k_h * \mathcal{X}_\alpha\|_{L^\infty(0,1-(\alpha+h))} + \|k_h\|_{L^\infty(0,1-(\alpha+h))}\\
&\leq \left\|\frac{T_{\alpha+h}k}{a_{\alpha+h}}\right\|_{L^\infty(0,1-(\alpha+h))} \int_0^t \left|\bar{\mathcal{X}}_h(s)\right| ds + \frac{1}{C}\|k_h\|_{L^\infty(0,1-(\alpha+h))}.
\end{aligned}$$

Using Gronwall's inequality, it follows that

$$\left\|\bar{\mathcal{X}}_h\right\|_{L^1(0,1-(\alpha+h))} \leq \frac{1}{C}\|k_h\|_{L^\infty(0,1-(\alpha+h))} \exp\left(\left\|\frac{T_{\alpha+h}k}{a_{\alpha+h}}\right\|_{L^\infty(0,1-(\alpha+h))}\right). \tag{3.19}$$

In view of (3.17) and (3.19), it suffices to show that for all $g \in X$, the quantities $\|T_{\alpha+h}g - T_\alpha g\|_{L^p(0,1-(\alpha+h))}$ and $\|k_h\|_{L^\infty(0,1-(\alpha+h))}$ both tend to zero as $h \to 0$ while boundedness of $\|T_{\alpha+h}k/a_{\alpha+h}\|_{L^\infty(0,1-(\alpha+h))}$ follows from continuity established in the arguments below.

Observe that if $\varphi(\cdot) : [0,\bar{\alpha}] \mapsto \mathbb{R}$ is a continuous function, then

$$\begin{aligned}
&\left| \int_{[0,\alpha+h]} \varphi(\rho) d\eta_{\alpha+h}(\rho) - \int_{[0,\alpha]} \varphi(\rho)\eta_\alpha(\rho) \right| \qquad (3.20)\\
&\le \left| \int_{[0,\alpha+h]} \varphi(\rho) d(\eta_{\alpha+h} - \eta_\alpha)(\rho) \right| + \left| \int_{(\alpha,\alpha+h]} \varphi(\rho) d\eta_\alpha(\rho) \right| \\
&\le \|\varphi\|_{C[0,\bar{\alpha}]} \left[\| |\eta_{\alpha+h} - \eta_\alpha| \| + |\eta_\alpha| \left((\alpha, \alpha+h] \right) \right]
\end{aligned}$$

which goes to zero as $h \to 0$ because $\lim_{h\to 0} |\eta_\alpha| \left((\alpha, \alpha+h] \right) = 0$ under the assumptions on η_α. In particular, the mappings $\alpha \mapsto \eta_\alpha([0,\alpha])$ and $\alpha \mapsto a_\alpha$ are continuous by taking $\varphi(\rho) \equiv 1$ and $\varphi(\rho) = \int_0^\rho k(s)\,ds$ respectively in (3.20).

Similarly, let $\varphi \in X$ and define $\varphi_{ext} \in L^p(0, 1+\bar{\alpha})$ by

$$\varphi_{ext}(t) := \begin{cases} \varphi(t) & \text{a.e. } t \in (0,1) \\ 0 & \text{otherwise.} \end{cases} \qquad (3.21)$$

Then Minkowski's integral inequality yields

$$\begin{aligned}
&\left\| \int_{[0,\alpha+h]} \varphi(\cdot+\rho) d\eta_{\alpha+h}(\rho) - \int_{[0,\alpha]} \varphi(\cdot+\rho) d\eta_\alpha(\rho) \right\|_{L^p(0,1-(\alpha+h))} \qquad (3.22)\\
&\le \left\| \int_{[0,\alpha+h]} \varphi_{ext}(\cdot+\rho) d(\eta_{\alpha+h} - \eta_\alpha)(\rho) \right\| + \left\| \int_{(\alpha,\alpha+h]} \varphi_{ext}(\cdot+\rho) d\eta_\alpha(\rho) \right\| \\
&\le \|\varphi\| \left[\| |\eta_{\alpha+h} - \eta_\alpha| \| + |\eta_\alpha| \left((\alpha, \alpha+h] \right) \right].
\end{aligned}$$

which also approaches zero as $h \to 0$ again by the assumptions on η_α.

It follows from the definitions of T_α and k_h together with the convergence established for the quantities in (3.20) and (3.22) that $\|T_{\alpha+h}g - T_\alpha g\|_{L^p(0,1-(\alpha+h))}$ and $\|k_h\|_{L^\infty(0,1-(\alpha+h))}$ tend to zero as desired proving continuity from the right. Continuity from the left at each $\alpha \in (0,\bar{\alpha}]$ is established by reversing the roles of α and $\alpha+h$ in the above arguments with $h < 0$ such that $(\alpha+h) \in (0,\bar{\alpha})$. $\square$

Finally, we establish in the next lemma that the assumption on $\|T_\alpha f\|_\alpha$ needed in Theorem 2.9 and Corollary 2.10 holds automatically if $\bar{\alpha} > 0$ is sufficiently small. It follows then from Theorem 2.9 that there is $\alpha_\star(\delta, f^\delta)$ satisfying the discrepancy principle (2.9) for all $\delta > 0$ sufficiently small and any $f^\delta \in B_\delta(f)$, assuming that $b(\cdot)$ is scaled appropriately with respect to τ (i.e., according to (2.11)). But even when δ is not small, we provide a condition under which existence of $\alpha_\star(\delta, f^\delta)$ is still assured, this time by appealing to Lemma 2.8 and provided again that $b(\cdot)$ is scaled appropriately.

Lemma 3.7. *There exists an* $\epsilon = \epsilon(f) > 0$ *for which* $\|T_\alpha f\|_{L^p(0,1-\alpha)} \geq \epsilon$ *for all* $\alpha \in (0,\bar{\alpha}]$ *and* $\bar{\alpha} = \bar{\alpha}(f) > 0$ *sufficiently small. Moreover, if* $\left\|f^\delta\right\|/\delta$ *is sufficiently large , then* $\left\|T_{\bar{\alpha}} f^\delta\right\|_{L^p(0,1-\bar{\alpha})} > \delta$.

Proof. We first establish that for every $g \in X$, the operators $\{T_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ satisfy

$$\|T_\alpha g - g\|_{L^p(0,1-\alpha)} \to 0 \quad \text{as} \quad \alpha \to 0. \tag{3.23}$$

To that end, let $g \in X$ and let g_{ext} be its extension to $[0, 1 + \bar{\alpha}]$ as in (3.21). The properties of local-regularizing measures, (3.8), (3.9), and Minkowski's inequality for integrals yield

$$\begin{aligned}
\|T_\alpha g- \ g\|_{L^p(0,1-\alpha)}& \\
&= \frac{1}{\eta_\alpha([0,\alpha])}\left\|\int_{[0,\alpha]} g_{ext}(\cdot+\rho) - g_{ext}(\cdot)d\eta_\alpha(\rho)\right\|_{L^p(0,1-\alpha)} \\
&\leq \frac{1}{\alpha^\sigma c_0(1-\bar{C}_0\bar{\alpha})}\int_{[0,\alpha]} \|g_{ext}(\cdot+\rho) - g_{ext}(\cdot)\|_{L^p(0,1-\alpha)}\, d\left|\eta_\alpha\right|(\rho) \\
&\leq M_T \sup_{\rho\in[0,\alpha]} \|g_{ext}(\cdot+\rho) - g_{ext}(\cdot)\|,
\end{aligned}$$

which tends to zero as $\alpha \to 0$ by continuity of translations in $L^p(0, 1 + \bar{\alpha})$, $p \in [1,\infty)$.

Fix $\hat{\alpha} \in (0, 1)$ and choose $c > 1$ so that

$$\frac{\|f\|}{c} \leq \|f\|_{L^p(0,1-\hat{\alpha})}$$

holds. From (3.23), we may choose $\bar{\alpha} = \bar{\alpha}(f) \in (0, \hat{\alpha}]$ sufficiently small so that for all $\alpha \in (0, \bar{\alpha}]$,

$$\|T_\alpha f - f\|_{L^p(0,1-\alpha)} \leq \frac{\|f\|_{L^p(0,1-\hat{\alpha})}}{c}.$$

Since $\|f\|_{L^p(0,1-\hat{\alpha})} \leq \|f\|_{L^p(0,1-\bar{\alpha})}$, it follows that

$$\|T_\alpha f - f\|_{L^p(0,1-\alpha)} \leq \frac{\|f\|_{L^p(0,1-\bar{\alpha})}}{c} \leq \frac{\|f\|}{c} \leq \|f\|_{L^p(0,1-\bar{\alpha})}$$

holds for all $\alpha \in (0, \bar{\alpha}]$. Thus

$$\begin{aligned}
\|T_\alpha f\|_{L^p(0,1-\alpha)} &\geq \|f\|_{L^p(0,1-\bar{\alpha})} - \|T_\alpha f - f\|_{L^p(0,1-\alpha)} \qquad (3.24)\\
&\geq \frac{c-1}{c}\|f\|_{L^p(0,1-\bar{\alpha})} \\
&\geq \frac{c-1}{c^2}\|f\|
\end{aligned}$$

and hence $\|T_\alpha f\|_{L^p(0,1-\alpha)} \geq \epsilon > 0$ for all $\alpha \in (0, \bar{\alpha}(f)]$, where $\epsilon = (c-1)\,\|f\|/c^2$.

If $\left\|f^\delta\right\|/\delta > 1 + c^2(1+M_T)/(c-1)$, then with (3.24) we have

$$\begin{aligned}
\left\|T_{\bar{\alpha}} f^\delta\right\|_{L^p(0,1-\bar{\alpha})} &\geq \|T_{\bar{\alpha}} f\|_{L^p(0,1-\bar{\alpha})} - \left\|T_{\bar{\alpha}} f^\delta - T_{\bar{\alpha}} f\right\|_{L^p(0,1-\bar{\alpha})} \\
&\geq \frac{c-1}{c^2}\,\|f\| - M_T\delta \\
&\geq \frac{c-1}{c^2}\left[\left\|f^\delta\right\| - \delta\right] - M_T\delta \\
&> \delta.
\end{aligned}$$

□

3.3 Convergence Results

We conclude with the main convergence results.

Recall that $X = L^p(0,1)$ for fixed $p \in (1,\infty)$ and $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]} \subseteq \mathcal{M}$ is a continuous local-regularizing family of measures such that for each $\alpha \in (0,\bar{\alpha}]$, η_α is concentrated on $[0,\alpha)$ or $\eta_\alpha(\{\alpha\}) = 0$. The operators $\{\mathcal{R}_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ are the corresponding local regularization operators in (2.3) with the assignments of X_α, r_α, T_α, $\mathcal{A}_\alpha$, and a_α made in (3.2)–(3.7), respectively, and $c(\alpha)$ in (3.10). Suppose the that data $\left\{f^\delta\right\}_{\delta>0}$, $f^\delta \in B_\delta(f)$, are given.

Theorem 3.8. *If $\bar{\alpha}$ is sufficiently small, then there exists $\alpha_\star = \alpha_\star(\delta, f^\delta) > 0$ which satisfies (2.9), $\lim_{\delta\to 0}\alpha_\star(\delta, f^\delta) = 0$ and*

$$\left\|u^\delta_{\alpha_\star} - \bar{u}\right\|_{L^p(0,1-\alpha_\star)} \to 0 \quad \text{as} \quad \delta \to 0. \tag{3.25}$$

Moreover, suppose that $\bar{u}$ satisfies the source condition $\bar{u} \in D(\mu)$, for some $\mu \in (0, \nu+1]$, where $D(\mu)$ is defined in (3.11), then with $b(\alpha) = \hat{C}\alpha^m$ for some $m, \hat{C} = \hat{C}(\bar{\alpha}, \tau) > 0$ in (2.10), there exists $\alpha_\star = \alpha_\star(\delta, f^\delta) > 0$ which satisfies the discrepancy criterion (2.9) and yields the rate of convergence

$$\left\|u^\delta_{\alpha_\star} - \bar{u}\right\|_{L^p(0,1-\alpha_\star)} = \mathcal{O}\left(\delta^{\zeta/(m+\nu)}\right) \quad \text{as} \quad \delta \to 0, \tag{3.26}$$

where $\zeta = \min\{m, \mu\}$.

Proof. Existence of $\alpha_\star$ and convergence in (3.25) are direct consequences of Lemmas 3.3, 3.6, and 3.7, and Theorem 2.9(i). The convergence rate in (3.26) for $\bar{u} \in D(\mu)$ follows from Theorem 2.9(ii) using $\omega(\alpha)$ as given in (3.12). □

4 Numerical Example

Numerical examples found in [2] illustrate the application of local regularization without the use of a discrepancy principle for solving the one–smoothing problem with $k(t) = e^{-t/2}$, the exact solution $\bar{u}$ given by

$$\bar{u}(t) = \begin{cases} -20t/3 + 1, & 0 \leq t \leq 0.3, \\ 5t - 2.5, & 0.3 < t \leq 0.5, \\ -5t + 2.5, & 0.5 < t \leq 0.7, \\ 20t/3 - 17/3, & 0.7 < t \leq 1, \end{cases}$$

and 3% relative error in the data.[6] For each $\alpha \in (0, \bar{\alpha}]$, T_α, $\mathcal{A}_\alpha$, and a_α are defined as in (3.5)–(3.7), where η_α is the continuous local-regularizing measure defined in Proposition 3.5 with $\psi(\rho) = -14.2776\rho + 12.0051$ stably constructed from $p_1(\lambda) = \lambda + 5$ with small parameter 0.001.[7] Note that both Lebesgue measure and a discrete measure are also valid choices for mildly smoothing Volterra problems (such as this example).[8]

For comparison and to illustrate practical use of the method described in Section 3, we revisit this example employing the same local regularization and collocation scheme previously used, however here the values of the local regularization parameter are selected using both the new modified discrepancy principle and the classical discrepancy principle (with the $\|\cdot\|$–norm measured on the reduced interval $[0, 1 - \alpha]$).

That is, $N = 200$ and $t_i = i/N$, $i = 1, \ldots, N$ are the equally spaced collocation points. The exact data, $f = \mathcal{A}\bar{u}$, is represented by the vector $f_N = (f(t_1), \ldots, f(t_N))^\top \in \mathbb{R}^N$. A uniformly distributed random error vector δ_N is added to f_N to form the noisy data vector f_N^δ. The absolute error $\delta = \|\delta_N\|_{\mathbb{R}^N} = 0.0043$ and the relative error in the data is $\delta / \|f_N\|_{\mathbb{R}^N} = 0.03$, where $\|\cdot\|_{\mathbb{R}^N}$ denotes the Euclidean norm on $\mathbb{R}^N$.

We fix $\bar{\alpha} = 0.08$ and $\tau = \sqrt{2}$. For $i = 1, \ldots, N$, let $\chi_{(t_{i-1}, t_i]}$ denote the usual characteristic function of the interval $(t_{i-1}, t_i]$ and $S_N = \text{span}\{\chi_{(t_{i-1}, t_i]}\}_{i=1}^N$. Note that we do not assume additional data is available beyond the interval $[0, 1]$ as would be needed to produce an accurate reconstruction on the entire interval with any method due to the nature of the Volterra problem. Thus, the local regularized approximation is that $v \in S_{N-r}$ which satisfies $(a_\alpha I + \mathcal{A}_\alpha)v = T_\alpha f^\delta$ at t_i, $i =$

[6] A three-smoothing problem was also treated in [2]

[7] Example 4.3 of [2] illustrates the stable construction of ψ associated with $p_\nu(\lambda)$ in (3.4) with small parameter β.

[8] Discrete measures also work in practice for the infinitely smoothing Inverse Heat Conduction Problem.

$1, \ldots, N - r$, where α is selected from the set $\Delta_{N,\bar{\alpha}} = \{r/N \mid r = 1, 2, \ldots 16\}$. We denote by $u^{\delta}_{N-r,\alpha} \in \mathbb{R}^{N-r}$, respectively $\bar{u}_{N-r} \in \mathbb{R}^{N-r}$, the vector with i^{th} component given by the collocation-based regularized solution v, respectively $\bar{u}$, at $t = ((i - .5)/N), i = 1, \ldots, N - r$. The relative solution error is given by $\left\| u^{\delta}_{N-r,\alpha} - \bar{u}_{N-r} \right\|_{\mathbb{R}^{N-r}} / \left\| \bar{u}_{N-r} \right\|_{\mathbb{R}^{N-r}}$.

The value of the new modified discrepancy functional is computed using $b(\alpha) = \eta_\alpha([0, \alpha]) = 4.866\alpha$ times the Euclidean norm of the discrete representation of $\mathcal{A}_\alpha u^{\delta}_{\alpha} - T_\alpha f^{\delta}$. Similarly, the classical discrepancy functional is the Euclidean norm of the discrete representation of $\mathcal{A} u^{\delta}_{\alpha} - f^{\delta}$. In this example, both the new modified discrepancy functional and classical discrepancy functional are observed to be monotone increasing in α on $[0, 1 - \alpha]$ hence the values of α are chosen as the greatest in $\Delta_{N,\bar{\alpha}}$ so that the discrepancy functionals do not exceed $\tau\delta$.

The new modified discrepancy principle predicts the value $\alpha_\star = 0.07$ and the relative solution error in $u^{\delta}_{186,\alpha_\star}$ is 13.4%. The classical discrepancy principle predicts the much smaller value of $\alpha_{DP} = 0.02$. To compare, the relative solution error in $u^{\delta}_{186,\alpha_{DP}}$ is 63.2%. The reconstructions are plotted on the interval $[0, 0.93]$ in Figure 1 with $\bar{u}$ displayed in dashed lines.

Furthermore, we illustrate in Table 1 and Figure 2 the asymptotic behavior of $\alpha_\star(\delta)$ using the new modified discrepancy principle and compare it to the theoretical result of $\delta^{1/2}$ in Theorem 3.8 (with $\nu = m = \mu = 1$).

δ	$\delta / \left\| f_N \right\|_{\mathbb{R}^N}$	$\alpha_\star$	$\alpha_\star / \delta^{1/2}$
0.00426	0.0300	0.08	1.23
0.00202	0.00143	0.06	1.33
0.00113	0.000795	0.45	1.34
0.000538	0.000379	0.025	1.08
0.000297	0.000210	0.020	1.16

Table 1. Asymptotic behavior of $\alpha_\star$.

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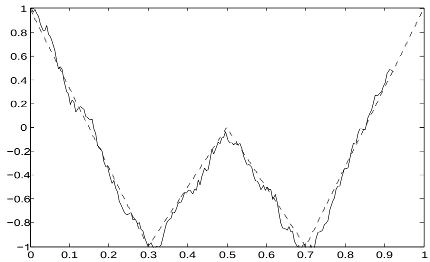
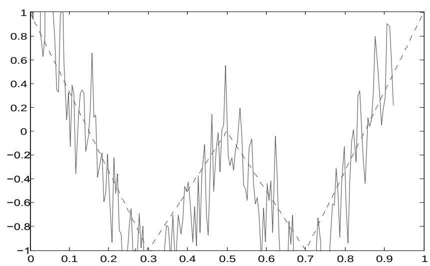


(a) Local regularized solution with regularization parameter $\alpha_\star = 0.07$ using the new modified discrepancy principle.



(b) Local regularized solution with regularization parameter $\alpha_{DP} = 0.02$ using the classical discrepancy principle.

Figure 1

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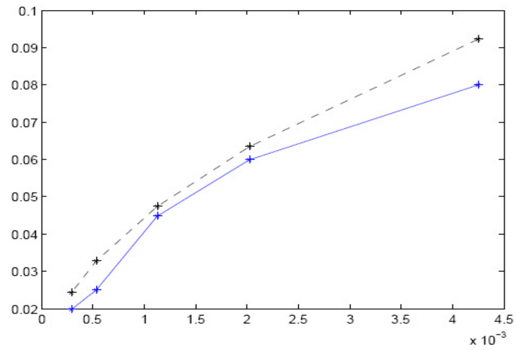

Figure 2. Local regularization parameter $\alpha_\star(\delta)$ (solid line) and $\delta^{1/2}$ (dashed line).

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